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LETTER TO THE EDITOR

# Dispersive water-wave equations: a paradigm of the Painlevé conjecture

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**Abstract.** We have analysed the equation of dispersive water waves within the formalism of a singular point analysis of the Painlevé conjecture. It is interesting to observe that the two different similarity reductions lead to apparently different-looking ordinary differential equations with a similar singularity structure. While one is a pure Painlevé equation, the other is not. On the other hand, a direct analysis of the PDE itself shows resonances at $r=-1, 2, 3, 4$ which contain those of the ODE. If, however, we adopt the exact procedure of Weiss *et al* and try to truncate the expansion then it is seen that the set of equations obtained are not overdetermined. Thus it is not possible to derive the Lax pair via the Painlevé analysis but a Backlund transformation can be written. The number of arbitrary coefficients is seen to be in conformity with the Cauchy–Kowalevskaya theorem.

Very recently the equations of dispersive water waves were analysed by Kupershmidt [1] within the formalism of Gel'fand and Dorfman [2] in relation to its multi-Hamiltonian structure. It was shown that there actually exists an infinite number of commutative Hamiltonian flows corresponding to the infinite number of conservation laws. This equation describing dispersive waves was solved by the IST procedure by Kaup and by Jaulent and Jean [3]. As is known, the existence of such properties is indicative of the complete integrability of such systems of equations (though there are some examples where this is not true). Here we have performed a singular-point analysis for a Painlevé test of this pair of equations.

At present there are three different approaches for the Painlevé analysis.

(i) In the first approach (as advocated by Ablowitz *et al* [4]) one uses the group-theoretic reduction to reduce the non-linear partial differential equations (NLPDE) to ordinary differential equations (ODE) which should be analysed regarding the pole structure. One should have all possible reductions of the PDE to ODE.

(ii) Next is the approach of Weiss *et al* [5] by which we seek the Painlevé property of the original NLPDE and it is suggested that if the number of equations for the unknown coefficients of expansion are very large and are overdetermined only then it will be possible to truncate the expansions at the constant level and to deduce the Lax pair.

(iii) Thirdly, we mention the simplified version of the approach due to Kruskal [6], where one determines only the resonance position and demonstrates the arbitrariness of the expansion coefficients in order to check the conformity with the Cauchy–Kowalevskaya theorem.

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In our present analysis we have here applied all these three approaches to obtain a decisive answer to the complete integrability of the set of equations under consideration. It is interesting to observe that although some common features arise from these various methods of analysis, they are neither completely identical nor do they contradict each other.

The equation pair is

$$\begin{aligned} u_t &= u_{xx} + h_x + uu_x \\ h_t &= -h_{xx} + hu_x + h_x u. \end{aligned} \tag{1}$$

Following Bluman and Cole [7] one can immediately determine the Lie point symmetries of (1) and hence observe that $(h, u)$ have the following similarity form:

$$h = \frac{1}{t} f\left(\frac{x^2}{t}\right) \qquad u = \frac{1}{x} g\left(\frac{x^2}{t}\right). \tag{2}$$

Then (1) reduces to

$$\begin{aligned} &2f' = -(1/z^2)(2g - g^2) + (2/Z)(g' - gg') - g' - 4g'' \\ &f + zf' + 2f' + 4zf'' + fg/z - 2(gf)' \end{aligned} \tag{2'}$$

where $f' = -f/z$, $g' = -g/z$ and $z = x^2/t$. The eliminant of (2) is

$$\begin{aligned} 16z^2 g''' = {}& g(z^2 + 12) - 3g^3 - zg^2 + 6zg^2 g' + gg'(4z + 6z^2) \\ & + g'(-16z + z^2 - 2Az^2) + zA - z^2 A. \end{aligned} \tag{3}$$

This equation cannot be integrated further, so it does not reduce to a second-order non-linear ordinary equation. Although ordinary non-linear equations of only second order have, in general, been classified and analysed from the point of view of Painlevé analysis, one can nevertheless follow the same methodology to study higher-order equations and prove similar theorems [8]. Very recently a non-linear ordinary equation of third order with properties similar to that of the Painlevé class has been studied by Martynov [9]. We therefore proceed here with a singular-point analysis of this third-order equation. We can, however, perform a singular-point analysis by setting

$$g \approx a_0 (z - z_0)^p$$

when

$$16zg''' \quad \text{and} \quad 6zg^1 g^2 \text{ matches with } p = -1, \ldots. \tag{4}$$

It is to be kept in mind that for the matching we have expanded each $z$-dependent coefficient in the neighbourhood of $z_0$ as

$$\begin{aligned} z^3 &\doteq (z - z_0 + z_0)^3 \\ &= (z - z_0) + 3z_0^2 (z - z_0) + 3z_0 (z - z_0)^2 + z_0^3 \end{aligned}$$

etc.

Actually (3) can be written as

$$\begin{aligned} 16[&(z - z_0)^3 + 3(z - z_0)^2 z_0 + 3(z - z_0) z_0^2 + z_0^3] g'' \\ &= g[(z - z_0) + 2z_0 (z - z_0) + z_0^2 + 12] - 3g^3 \\ &\quad + g'[(z - z_0)^3 + 3z_0 (z - z_0)^2 + 3z^2 (z - z_0) + z_0^3 - 12(z - z_0) - 12z_0] \\ &\quad - 2g^2 + 6gg'[(z - z_0)^2 + 2z_0 (z - z_0) + z_0^2] + 6g' g^2 [(z - z_0) + z_0]. \end{aligned} \tag{5}$$

To determine the resonance we set

$$g \approx a_0 (z - z_0)^{-1} + a_0 (z - z_0)^{p-1} \tag{6}$$

whence we get $a_0 = 4z_0$ and comparing the coefficients of $(z-z_0)^{p-4}$ we observe that $p$ satisfies an equation of the form

$$p(p-1)(p-5)+12=0 \qquad \text{or} \qquad (p+1)(p-4)(p-3)=0. \tag{7}$$

So we have resonances at $p=-1, 3, 4$. It is not difficult to proceed further and check the arbitrariness of the expansions at these resonances so that for a third-order equation if we get three resonances and three arbitrary constants, we certainly satisfy the Cauchy-Kowalevskaya criterion. We now observe that our equation (3) has satisfied the Cauchy-Kowalevskaya condition so it may be possible by suitable and intricate manipulation to reduce it to one of the second-order ODE of Painlevé type according to the method of [10] in which Bureau discusses both third-order and second-order ODE of Painlevé type, but in our case it is not very apparent. On the other hand, it is really not surprising that a third-order equation possess the requisite properties (see [9]).

Another reduction that follows from the translation invariance is

$$h=h(x-vt) \qquad u=u(x-vt) \qquad z=x-vt. \tag{8}$$

Then we obtain

$$\begin{aligned} -vu' &= u''+h'+uu' \\ -vh' &= -h''+(uh)' \end{aligned} \tag{9}$$

and again if we eliminate $h$ we obtain

$$\mathrm{d}^2u/\mathrm{d}z^2 - v^2u + \tfrac{3}{2}u^2 - \tfrac{1}{2}u^3 = 0 \tag{10}$$

where we have chosen the constant of integration to be zero. Equation (10) is similar to the Painlevé equation discussed by Ince [8] and can actually be solved by elliptic functions. On the other hand, we have by singularity analysis $u \approx a_0(z-z_0)^{-1}$ with resonances at $r=-1, 4$, which is a subclass of both the sets given by (7) and (8). Note also that (10) is a second-order equation.

Next we turn to the direct analysis of the PDE by a method originally suggested by Weiss and later simplified by Kruskal. Here, after the determination of the leaading exponents we set

$$\begin{aligned} u &= \sum_{j=0}^{\infty} a_j(t)\phi^{j-\gamma}(x,t) \\ h &= \sum_{j=0}^{\infty} b_j(t)\phi^{j-\gamma}(x,t). \end{aligned} \tag{11}$$

For the determination of the leading exponents we set $u \approx a_0(t)\phi^{\alpha}(x,t)$, $h \approx b_0(t)\phi^{\beta}(x,t)$. Then we observe that two cases may arise:

(I) $\alpha=-1$, $\beta=-2$ for which $a_0=-2$ and $b_0=-4$, and
(II) $\alpha=-1$, $\beta=1,2$ for which $a_0=2$ and $b_0$ is not determined.

Therefore in case I we set

$$u = \sum_{j=0}^{\infty} a_j(t)\phi^{j-1}(x,t) \qquad h = \sum_{j=0}^{\infty} b_j(t)\phi^{j-2}(x,t).$$

The recursion relations obtained are

$$a_{j-2,t}+a_{j-1}(j-2)\phi_t=(j-1)(j-2)a_j+(j-2)b_j+\sum_{k=0}^{i} a_{j-k}a_k(k-1) \tag{12}$$

$$b_{j-2,t}+(j-3)b_{j-1}\phi_t+(j-2)(j-3)b_j=\sum_{k=0}^{i} a_{j-k}b_k(j-3). \tag{13}$$

From these relations it is not difficult to guess that the equations leading to the resonance positions are

$$(r+1)(r-2)(r-3)(r-4)=0. \tag{14}$$

Thus we get resonances at $r=-1,2,3,4$. The resonance at $r=-1$ corresponds to the arbitrariness of $\phi$. We now proceed to check the coefficients at $r=2,3,4$ via (9) and (10):

for $j=0$ $$b_0=-4 \tag{15a}$$

$$a_0=-2 \tag{15b}$$

for $j=1$ $$a_1=\phi_t \qquad b_1=0 \tag{15c}$$

for $j=2$ $$\begin{aligned} &a_{0t}=0 \quad \text{(identically satisfied)}\\ &b_2=-2a_2 \end{aligned} \tag{16}$$

for $j=3$ $$\begin{aligned} &b_3=\phi_{tt}\\ &b_{1t}=0 \end{aligned} \tag{17}$$

for $j=4$ $$\begin{aligned} &a_{2t}-a_2^2=2b_4+2a_4\\ &b_{2t}+a_2b_2+4a_4+2b_4=0. \end{aligned} \tag{18}$$

Now using (17) and (18) we observe that the second equation of (18) is

$$4a_4+2b_4=2a_{2t}+2a_2^2 \qquad \text{or} \qquad 2a_4+b_4=a_{2t}+a_2^2.$$

Together with (16) this implies that

$$b_4=-2a_2^2 \qquad \text{and} \qquad 2a_4=a_{2t}+3a_2^2 \tag{19}$$

so that the coefficients at the resonance positions $r=2,3,4$ are undetermined, and it seems that we may conclude that the equation pair is completely integrable.

In case II we set identically

$$\begin{aligned} &u\approx a_0\phi^{-1}+a_1\phi^{p-1}\\ &h\approx b_0\phi+b_1\phi^{p+1}. \end{aligned} \tag{20}$$

After substitution and equating coefficients of $\phi^{p-1}$ we obtain

$$\begin{aligned} &a_1b_0p+b_1p(1-p)=0\\ &a(p-1)(p-2)+2(p-2)=0 \end{aligned} \tag{21}$$

so that the system matrix is

$$\begin{bmatrix} b_0p & p(1-p)\\ (p-1)(p-2)+2(p-2) & 0 \end{bmatrix}=0 \tag{22}$$

leading to $p=0$, $-1$, 1, 2, so we again get four resonances including that at $p=-1$. Arbitrariness of the coefficients can be checked as previously. Note that in case II we could also set

$$\begin{aligned} u &\approx a_0\phi^{-1}+a_1\phi^{p-1} \\ h &\approx b_0\phi^2+b_1\phi^{p+2}. \end{aligned} \tag{23}$$

The resonance positions are then seen to be at

$$p=0,-1,-1,2. \tag{24}$$

Having a double resonance at $p=-1$, we lose one arbitrariness and this branch does not pass the Painlevé test. In the former class of this case, i.e. $\alpha=+1$, $\beta=+1$, we set out to ascertain the arbitrariness of the coefficients. We set

$$\begin{aligned} u &= \sum a_j(t)\phi^{j-1}(x,t) \\ h &= \sum b_j(t)\phi^{j+1}(x,t). \end{aligned} \tag{25}$$

The recursion relations are

$$\begin{aligned} &a_{j-2,t}+a_{j-1}(j-2)\phi_t = a_j(j-1)(j-2)\phi_x^2+b_{j-3}(j-2)\phi_x+\sum_{k=0}^{j} a_{j-k}a_k(k-1)\phi_x \\ &b_{j-3,t}+jb_{j-1}\phi_t = b_j(j+1)j\phi_x^2+\sum_{k=0}^{j} b_{j-k}(k-1)a_k\phi_x+\sum_{k=0}^{j} a_{j-k}b_k(k+1)\phi_x. \end{aligned} \tag{26}$$

From these we easily obtain $a_0=2$, $a_1=\phi_t$, $b_1=0$, $\phi_{tt}=4a_{23}+b_0$ and $b_{0t}=10b_2+2b_0a_2$, so that $b_{00}$, $a_1$, $a_2$, $b_2$ are all arbitrary and we again can satisfy the Cauchy-Kowalevskaya condition.

Let us now modify and extend the above calculations further and try to apply the full machinery of Weiss *et al* [5] without any simplification. That is, we now set

$$\begin{aligned} u &= \sum_{j=0}^{\infty} a_j(x,t)\phi^{j-1}(x,t) \\ h &= \sum_{j=0}^{\infty} b_j(x,t)\phi^{j-2}(x,t). \end{aligned} \tag{27}$$

Substituting in (1) and equating respectively coefficients of $\phi^{j-3}$ and $\phi^{j-4}$ we obtain

$$\begin{aligned} a_{j-2,t}&+(j-2)a_{j-1}\phi_t \\ &= a_{j-2,xx}+2(j-2)a_{j-1,x}\phi_x+(j-2)a_{j-1}\phi_{xx}+\sum_{k=0}^{j} a_{j-k}a_{kx} \\ &\quad+\sum_{k=0}^{j} a_{j-k}a_k(k-1)\phi_x+(j-1)(j-2)a_j\phi_x^2+b_{j-1,x}+(j-2)b_j\phi_x \end{aligned} \tag{28a}$$

$$\begin{aligned} b_{j-2,t}&+(j-3)b_{j-1}\phi_t+b_{j-2,xx}+2(j-3)b_{j-1,x}\phi_x+(j-2)(j-3)b_j\phi_x^2+(j-3)b_{j-1}\phi_{xx} \\ &= \sum_{k=0}^{j} a_{j-k-1,x}b_x+\sum_{k=0}^{j} a_{j-k-1}b_{xx}+\sum_{k=0}^{j} a_{j-k}b_k(j-3)\phi_x. \end{aligned} \tag{28b}$$

Resonance positions are determined as before to be at $r=-1,2,3,4$. We now proceed to check the coefficients at the resonance positions:

$$\text{for } k=0 \qquad \begin{aligned} a_0 &= -2\phi_x \\ b_0 &= -4\phi_x^2 \end{aligned} \tag{29}$$

for $j=1$
$$a_1\phi_x=\phi_t+\phi_{xx}$$
$$b_1=4\phi_{xx}. \tag{30}$$

For $j=2$ before writing down the equations we make the following observations. In the formalism of Weiss we usually cut off the series expansions (18) at a constant level, i.e. we demand

$$u=a_0\phi^{-1}+a_1$$
$$h=b_0\phi^{-2}+b_1\phi^{-1}+b_2 \tag{31}$$

by setting $a_i=0$, $i\geqslant 2$, and $b_j=0$, $j\geqslant 3$. If we use these then

for $j=2$
$$\phi_x^2b_2=-2\phi_x\phi_{xt}+2\phi_t\phi_{xx}+2\phi_{xx}^2-2\phi_x\phi_{xxx} \tag{32}$$

and the other equation at $j=2$ is identically satisfied.

For $j=3$
$$b_{1t}+b_{1xx}=a_{1x}b_1+a_{0x}b_2+a_1b_{1x}+a_0b_{2x} \tag{33a}$$
$$a_{1t}=a_{1xx}+b_{2x}+a_1a_{1x} \tag{33b}$$

(which is the equation for $u$).

For $j=4$
$$b_2=-b_{2xx}+(a_1b_2)_x \tag{34}$$

(which is the equation for $h$) and the other equation is an identity. Due to truncation we must prove the consistency of (22), (24) and the first equation of (25).

From (24) differentiating with respect to $x$ we have

$$\phi_xb_2=-2\phi_{xt}+\frac{2\phi_{xt}}{\phi x}\phi_{xx}+\frac{2\phi_{xx}^2}{\phi x}-2\phi_{xxx}. \tag{35}$$

Using (29) for $b_1$ and $b_2$ we observe from (35) that $\phi$ satisfies the equation

$$\phi_{tx}+\phi_{xxx}=0 \qquad \text{or} \qquad \phi_t+\phi_{xx}=c(t) \tag{36}$$

which is nothing but the diffusion-type linear equation. If we consider a special case where the constant $c$ is zero then the most general solution of (36) can be written as

$$\phi=F(t)\exp(x^2/4t).$$

Now let us apply (31) by starting with the trivial solution $a_1=b_2=0$. Then

$$u=(\log\phi)_x=x/2t$$
$$h=4(\partial/\partial x)(\phi_x/\phi)=2/t.$$

One can immediately verify that $u=x/2t$, $h=2/t$ is a solution of (1), so it is proved that (31) is a Bäcklund transformation when $\phi$ is a solution of (36). Thus by proceeding in the method of Weiss we have actually constructed a Bäcklund transformation for the dispersive water-wave problem.

In our above analysis we have presented a detailed analysis of the dispersive water-wave equation regarding its complete integrability. In each method we have determined the resonance positions and have checked the arbitrariness of the coefficients theorem. It is an interesting outcome of our analysis that even when a Lax pair cannot be obtained (in the case of a coupled system it is not known how to derive a Law pair from Painlevé analysis), a Bäcklund transformation can actually be constructed which generates non-trivial solutions from trivial ones. We have obtained a third-order ODE which satisfies the Painlevé property. The third-order ODE of Painlevé type has, however, been studied previously by Bureau [10].

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